

Separable states and positive maps

Erling Størmer

11-10-2007

Abstract

Using the natural duality between linear functionals on tensor products of C^* -algebras with the trace class operators on a Hilbert space H and linear maps of the C^* -algebra into $B(H)$, we study the relationship between separability, entanglement and the Peres condition of states and positivity properties of the linear maps.

Introduction

In an earlier paper [14] we studied the duality between linear functionals $\tilde{\phi}$ on a tensor product $A \hat{\otimes} \mathcal{T}$ of an operator system A and the trace class operators \mathcal{T} on a Hilbert space H , and bounded linear maps $\phi: A \rightarrow B(H)$ given by the formula $\tilde{\phi}(a \otimes b) = \text{Tr}(\phi(a)b^t)$. The main emphasis was on positivity properties of $\tilde{\phi}$ on cones in $A \hat{\otimes} \mathcal{T}$ obtained by classes of positive maps. In the present paper we shall see how this study yields a natural framework for the study of separable states of $A \hat{\otimes} \mathcal{T}$, for example we recover results of Horodecki et.al [9] and Horodecki, Shor and Ruskai [11] on characterizations of separable states. In addition we shall obtain characterizations of states on $A \hat{\otimes} \mathcal{T}$ satisfying the Peres condition, viz $\rho \circ (\iota \otimes t)$ is positive, where t is the transpose map and ι the identity map. In particular we see that nondecomposable maps yield natural examples of entangled states which satisfy the Peres condition; for this see also [7], [8]. In the last section we study the definite set of a positive map ϕ on a C^* -algebra A , i.e. the set of self-adjoint operators in A such that $\phi(a^2) = \phi(a)^2$, and show that if $\tilde{\phi}$ is a separable state, then the image of the definite set is a C^* -subalgebra of the center of the C^* -algebra generated by $\phi(A)$. As a corollary we obtain a decomposition result for separable states in the finite dimensional case.

The author is indebted to E.Alfsen for his careful reading of the manuscript and several useful comments.

1 Cones and states

In this section we recall notation and concepts from [14] and show a general characterization of separable states close to that in [11]. For more details on the following see [14].

By an *operator system* we shall mean a norm closed self-adjoint set A of bounded operators on a Hilbert space containing the identity. We denote by $A \odot B(H)$ the algebraic tensor product of A and $B(H)$ and by $A \widehat{\otimes} B(H)$ the closure of $A \odot B(H)$ in the operator norm. If \mathcal{T} denotes the trace class operators on H , then $A \widehat{\otimes} \mathcal{T}$ denotes the projective tensor product of A and \mathcal{T} . We denote by $B(A, H)$, (resp. $B(A, H)^+$) the bounded (resp. positive) linear maps of A into $B(H)$. The *BW-topology* on $B(A, H)$ is the topology of bounded pointwise weak convergence, i.e. a net (ϕ_ν) converges to ϕ if it is uniformly bounded, and $\phi_\nu(a) \rightarrow \phi(a)$ weakly for all $a \in A$. We denote by t the transpose map on $B(H)$ with respect to some orthonormal basis for H . Then by abuse of notation we get that the transpose map on $B(K) \otimes B(H)$ is $t \otimes t$. We shall also denote by Tr the usual trace on $B(H)$ which takes the value 1 on minimal projections. We recall Lemma 2.1 in [14].

Lemma 1 *With the above notation there is an isometric isomorphism $\phi \rightarrow \tilde{\phi}$ between $B(A, H)$ and $(A \widehat{\otimes} \mathcal{T})^*$ given by*

$$\tilde{\phi}(a \otimes b) = Tr(\phi(a)b^t), \quad a \in A, b \in \mathcal{T}.$$

Furthermore, $\phi \in B(A, H)^+$ if and only if $\tilde{\phi}$ is positive on the cone $A^+ \widehat{\otimes} \mathcal{T}^+$ generated by operators of the form $a \otimes b$ with a and b positive.

We recall Definition 2.3 in [14]. It says that a BW-closed subcone $K \neq 0$ of $B(B(H), H)^+$ is a *mapping cone* if it has a BW-dense subset of ultra weakly continuous maps and is invariant in the sense that if $\alpha \in K$, and $a, b \in B(H)$ then the map $x \rightarrow a\alpha(bxb^*)a^*$ belongs to K . Three mapping cones will be of special interest in the following, namely $B(B(H), H)^+$, $CP(H)$ - the set of completely positive maps in $B(B(H), H)$, and $S(H)$ - the BW-closed cone generated by maps of the form

$$x \rightarrow \sum_{i=1}^n \omega_i(x) a_i,$$

where ω_i is a normal state on $B(H)$ and $a_i \in B(H)^+$. The latter maps are said to be of "Holevo form" in [11]. By Lemma 2.4 in [14] $S(H)$ is the minimal mapping cone and $B(B(H), H)^+$ the maximal one.

If K is a mapping cone and A an operator system as before, we denote by $P(A, K)$ the cone

$$P(A, K) = \{x \in A \widehat{\otimes} \mathcal{T} : \iota \otimes \alpha(x) \geq 0 \quad \forall \alpha \in K\}.$$

By Lemma 2.8 in [14] $P(A, K)$ is a proper norm closed convex cone in $A \widehat{\otimes} \mathcal{T}$ containing the cone $A^+ \widehat{\otimes} \mathcal{T}^+$. A map $\phi \in B(A, H)$ is said to be *K-positive* if

$$\tilde{\phi}(\sum a_i \otimes b_i) = \sum Tr(\phi(a_i)b_i^t) \geq 0 \text{ whenever } \sum a_i \otimes b_i \in P(A, K).$$

By Theorem 3.2 in [14] ϕ is completely positive if and only if $\tilde{\phi}$ is positive on the cone $(A \widehat{\otimes} \mathcal{T})^+$, the closure of the positive operators in $A \odot \mathcal{T}$, if and only if ϕ is $CP(H)$ -positive.

If $C \subseteq V$ and $D \subseteq W$ are closed convex cones in two real locally convex vector spaces in duality, we denote by C^* (resp. D^*) the set of $w \in W$ such that $\langle v, w \rangle \geq 0 \forall v \in C$, (resp. $v \in V$ such that $\langle v, w \rangle \geq 0 \forall w \in D$). Thus ϕ is K-positive if and only if $\tilde{\phi} \in P(A, K)^*$. By a straightforward application of the Hahn-Banach Theorem for closed convex cones, see e.g. [1], Prop. 1.32, we have

$$P(A, K) = P(A, K)^{**}.$$

We say a positive linear functional ρ on $A \otimes B(H)$ is *separable* if it belongs to the norm closure of positive sums of states of the form $\sigma \otimes \omega$, where σ is a state of A and ω a normal state of $B(H)$. Otherwise ρ is called *entangled*. We denote the set of separable states by $S(A, H)$. It is a norm closed cone in $(A \widehat{\otimes} \mathcal{T})^*$. As for P above $S(A, H) = S(A, H)^{**}$. Our next result is closely related to Theorem 2 in [11].

Theorem 2 *Let A be an operator system and $\phi \in B(A, H)$. Then the following conditions are equivalent:*

- (i) $\tilde{\phi}$ is a separable positive linear functional.
- (ii) ϕ is $S(H)$ -positive.
- (iii) ϕ is a BW-limit of maps of the form $x \rightarrow \sum_{i=1}^n \omega_i(x) b_i$ with ω_i a state of A , and $b_i \in B(H)^+$.

Proof. (i) \Leftrightarrow (ii). Let S_n denote the positive normal linear functionals on $B(H)$, and let $x = \sum x_i \otimes y_i \in A \odot B(H)$. Then

$$\begin{aligned} x &\in P(A, S(H)) \\ &\Leftrightarrow (\iota \otimes b\omega)(x) \geq 0 \forall \omega \in S_n, b \geq 0 \\ &\Leftrightarrow \sum x_i \omega(y_i) \otimes b = \sum x_i \otimes \omega(y_i) b \geq 0 \forall \omega \in S_n, b \geq 0 \\ &\Leftrightarrow \sum x_i \omega(y_i) \geq 0 \forall \omega \in S_n \\ &\Leftrightarrow \rho \otimes \omega(x) = \sum \rho(x_i) \omega(y_i) = \rho(\sum x_i \omega(y_i)) \geq 0 \forall \omega \in S_n, \rho \in A^{*+} \\ &\Leftrightarrow \eta(x) \geq 0 \forall \eta \in S(A, H) \\ &\Leftrightarrow x \in S(A, H)^*. \end{aligned}$$

Thus ϕ is $S(H)$ -positive if and only if $\tilde{\phi} \in P(A, S(H))^* = S(A, H)^{**} = S(A, H)$, proving that (i) \Leftrightarrow (ii). The equivalence (ii) \Leftrightarrow (iii) follows from Theorem 3.6 in [14], since a map $\alpha \in S(H)$ if and only if $t \circ \alpha \circ t \in S(H)$. The proof is complete.

In [11] maps like $x \rightarrow \sum \omega_i(x) b_i$ are called "entanglement breaking".

It is possible to give a direct proof of a less general form of the equivalence (i) \Leftrightarrow (iii) above. Suppose $\phi(a) = \sum \omega_i(a) b_i$ for $a \in A, b_i \in B(H)^+, \omega_i$ state of A . Then

$$\tilde{\phi}(a \otimes b) = \text{Tr}(\phi(a) b^t) = \sum \text{Tr}(\omega_i(a) b_i b^t) = \sum \omega_i(a) \text{Tr}(b_i b^t) = \sum \omega_i(a) \rho_i(b),$$

where $\rho_i(b) = \text{Tr}(b_i b^t)$ is a positive linear functional. Thus $\tilde{\phi}$ is separable.

Conversely, if $\tilde{\phi} = \sum \omega_i \otimes \rho_i$, let $\tilde{\rho}_i(b) = \rho_i(b^t) = \text{Tr}(b_i b)$. Then we have

$$\text{Tr}(\phi(a)b^t) = \tilde{\phi}(a \otimes b) = \sum \omega_i(a)\rho_i(b) = \sum \omega_i(a)\tilde{\rho}_i(b^t) = \sum \text{Tr}(\omega_i(a)b_i b^t).$$

This holds for all $b \in \mathcal{T}$, hence $\phi(a) = \sum \omega_i(a)b_i$.

Corollary 3 *Let H be separable and $\phi \in B(A, H)^+$. Suppose $\phi(A)$ is contained in an abelian C^* -algebra. Then $\tilde{\phi}$ is separable.*

Proof. By hypothesis there is an abelian von Neumann algebra $B \subseteq B(H)$ such that $\phi: A \rightarrow B$. Let (B_n) be an increasing sequence of finite dimensional von Neumann subalgebras of B such that $\bigcup_n B_n$ is weakly dense in B . Let $E_n: B \rightarrow B_n$ be normal conditional expectations such that $E_{n-1}|_{B_n} \circ E_n = E_{n-1}$. Then $\phi(x) = \text{wlim}_n E_n \circ \phi(x)$ for all $x \in A$. Since B_n is finite dimensional, $E_n \circ \phi(x) = \sum \omega_i^n(x)e_i^n$, where ω_i^n are positive linear functionals on A and e_i^n are minimal projections in B_n . Since ϕ is a BW-limit of the sequence $E_n \circ \phi$, ϕ is separable by Theorem 2. The proof is complete.

A celebrated necessary condition for a state ρ on $A \hat{\otimes} \mathcal{T}$ to be separable is the *Peres condition*, i.e. $\rho \circ (\iota \otimes t) \geq 0$. A map $\phi \in B(A, H)$ is said to be *copositive* if $t \circ \phi$ is completely positive.

Proposition 4 *Let $\phi \in B(A, H)$. Then $\tilde{\phi}$ satisfies the Peres condition if and only if ϕ is both completely positive and copositive.*

Proof. If $a \in A$ and $b \in \mathcal{T}$ we have, since the trace is invariant under transposition,

$$\tilde{\phi}(a \otimes b^t) = \text{Tr}(\phi(a)b) = \text{Tr}(t \circ \phi(a)b^t) = (t\tilde{\phi})(a \otimes b).$$

Thus $\tilde{\phi}$ satisfies the Peres condition if and only if both $t\tilde{\phi}$ and $\tilde{\phi}$ are positive. Using Theorem 3.2 in [14] this holds if and only if $t \circ \phi$ and ϕ are completely positive, hence if and only if ϕ is both completely positive and copositive.

2 States on $B(K) \otimes B(H)$

In this section we study the case when the operator system A equals $B(K)$ for a Hilbert space K . But first we consider the finite dimensional case. Let $M_n = M_n(\mathbb{C})$ denote the complex $n \times n$ matrices, and let $\phi: M_n \rightarrow M_m$, so $\phi \in B(A, \mathbb{C}^m)$, where $A = M_n$ and $H = \mathbb{C}^m$. Let (e_{ij}) be a complete set of matrix units in M_n . Then the *Choi matrix* for ϕ is

$$C_\phi = \sum e_{ij} \otimes \phi(e_{ij}) = \iota \otimes \phi(P) \in M_n \otimes M_m,$$

where $\frac{1}{n}P$ is the 1-dimensional projection $\frac{1}{n} \sum e_{ij} \otimes e_{ij}$, - the so-called maximally entangled state, see [3]. Denote by ϕ^t the map $t \circ \phi \circ t$, where t denotes the

transpose map in either M_n or in M_m . Then ϕ is completely positive if and only if ϕ^t is completely positive. It was shown by Choi [3] that ϕ is completely positive if and only if C_ϕ is positive. We use the convention that the density matrix for a state ρ is the positive matrix h such that $\rho(x) = \text{Tr}(hx)$.

Lemma 5 C_{ϕ^t} is the density matrix for $\tilde{\phi}$.

Proof. Let $a \in M_n, b \in M_m$. Since the transpose t on $M_n \otimes M_m$ is the tensor product of the transpose operators on M_n and M_m , we have

$$\begin{aligned} \text{Tr}(C_{\phi^t} a \otimes b) &= \sum \text{Tr}(e_{ij} \otimes \phi^t(e_{ij})(a \otimes b)) \\ &= \sum \text{Tr}(e_{ji} \otimes \phi(e_{ij}^t)(a^t \otimes b^t)) \\ &= \sum \text{Tr}(e_{ji} a^t) \text{Tr}(\phi(e_{ji}) b^t) \\ &= \sum a_{ji} \text{Tr}(e_{ji} \phi^*(b^t)) \\ &= \text{Tr}(a \phi^*(b^t)) \\ &= \tilde{\phi}(a \otimes b). \end{aligned}$$

In the above computation ϕ^* is the adjoint of ϕ as an operator between M_n and M_m considered as the trace class operators on \mathbb{C}^n and \mathbb{C}^m respectively. The proof is complete.

Lemma 6 Let $H = \mathbb{C}^m$ and $\phi \in B(M_n, H)$. Then ϕ is positive if and only if $C_{\phi^t} \in P(M_n, S(H))$, if and only if $C_\phi \in P(M_n, S(H))$. Hence $P(M_n, S(H)) = \{C_\phi : \phi \geq 0\}$.

Proof. By Theorem 2, or rather the proof of the equivalence (i) \Leftrightarrow (ii),

$$\begin{aligned} C_{\phi^t} \in P(M_n, S(H)) &= S(M_n, H)^* \\ \Leftrightarrow \text{Tr}(C_{\phi^t} a \otimes b) &\geq 0 \quad \forall a \in M_n^+, b \in M_m^+ \\ \Leftrightarrow \phi &\geq 0 \end{aligned}$$

by Lemma 1, proving the first statement. Since $\phi \geq 0$ if and only if $\phi^t \geq 0$, the above is equivalent to C_ϕ being in $P(M_n, S(H))$.

Each element $x \in P(M_n, S(H))$ defines a linear functional ρ on $M_n \otimes M_m$ by $\rho(y) = \text{Tr}(xy)$. By Lemma 1 there is $\phi \in B(M_n, \mathbb{C}^m)$ such that $\rho(a \otimes b) = \text{Tr}(\phi(a)b^t)$, hence by Lemma 5 and the first part of the proof, $x = C_{\phi^t}$ with $\phi \geq 0$. Thus the last statement follows, completing the proof.

We shall now apply the finite dimensional results to study states on $B(K) \otimes B(H)$ and to prove an infinite dimensional extension of the Horodecki Theorem [9]. Recall that a state and a positive linear map on a Von Neumann algebra are said to be normal if they are weakly continuous on bounded sets.

Theorem 7 *Let ρ be a normal state on $B(K) \otimes B(H)$ with K and H Hilbert spaces and with density operator h . Then ρ is separable if and only if $\iota \otimes \psi(h) \geq 0$ for all normal positive maps $\psi: B(H) \rightarrow B(K)$.*

Proof. Suppose ρ is separable and normal. Then $\rho \circ (\iota \otimes \phi)$ is a normal state for all unital normal positive maps $\phi: B(K) \rightarrow B(H)$. Let ψ be as in the statement of the theorem. Then the adjoint map ψ^* is a positive map of the trace class operators on K into those on H . Thus if $x \geq 0$ is of finite rank in $B(K \otimes K) = B(K) \otimes B(K)$, then

$$\text{Tr}((\iota \otimes \psi)(h)x) = \text{Tr}(h(\iota \otimes \psi^*)(x)) = \rho(\iota \otimes \psi^*(x)) \geq 0,$$

hence $\iota \otimes \psi(h) \geq 0$.

To show the converse we first assume K and H are finite dimensional. Then by Lemma 6 $P(M_n, S(H)) = \{C_\phi : \phi \geq 0\}$. Thus by Theorem 2 and Lemma 5 ρ is separable if and only if for all positive $\phi: B(K) \rightarrow B(H)$

$$\text{Tr}((\iota \otimes \phi^*)(h)P) = \text{Tr}(h(\iota \otimes \phi)(P)) = \text{Tr}(hC_\phi) \geq 0,$$

where P is the rank one matrix such that $C_\phi = \iota \otimes \phi(P)$. Since $P \geq 0$, and by assumption $\iota \otimes \phi^*(h) \geq 0$, it follows that ρ is separable.

We next consider the general case when K and H may be infinite dimensional. Assume $\iota \otimes \psi(h) \geq 0$ for all normal $\psi: B(H) \rightarrow B(K)$. Since the maps $\psi_f(x) = \psi(fxf)$ are positive for all finite dimensional projections f , it is clear that $\iota \otimes \psi((e \otimes f)h(e \otimes f)) \geq 0$ for all normal positive maps $\psi: B(H) \rightarrow B(K)$ with e a finite dimensional projection in $B(K)$. Let

$$\psi_{e \otimes f}(y) = e\psi(fyf)e, \quad y \in B(H).$$

Then $\iota \otimes \psi_{e \otimes f}(h) \geq 0$. Now every normal positive map $\phi: B(fH) \rightarrow B(eK)$ is of the form $\psi_{e \otimes f}$ with ψ as above, because we can define $\phi: B(H) \rightarrow B(K)$ by $\psi(x) = \phi(fxf)$. Thus by the part of the proof on the finite dimensional case, the positive linear functional $\omega(x) = \rho((e \otimes f)x(e \otimes f))$ is separable on $B(eK) \otimes B(fH)$. Since this holds for all finite dimensional projections e and f and ρ is normal, it follows that ρ is separable. The proof is complete.

We expect that the above theorem can be generalized to Von Neumann algebras other than $B(K)$. If A is a semi-finite Von Neumann algebra then so is $A \otimes B(H)$, hence each normal state on $A \otimes B(H)$ has a density operator with respect to a trace, and the formulation of the theorem has a natural generalization. In the type III case a formulation in terms of modular theory ought to be possible.

We next restate the Peres condition in terms of the density matrix of the normal state ρ .

Theorem 8 *Let ρ be a normal state on $B(K) \otimes B(H)$ with density operator h , and let t denote the transpose map of either $B(K)$ or $B(H)$. Then the following*

conditions are equivalent:

- (i) ρ satisfies the Peres condition.
- (ii) $\iota \otimes t(h) \geq 0$.
- (iii) $t \otimes \iota(h) \geq 0$.
- (iv) $h \in P(B(K), CP(H)) \cap P(B(K), \text{copos}(H))$, where $\text{copos}(H)$ denotes the copositive maps of $B(H)$ into itself.

Proof. Assume (i). Since the trace on $B(K) \otimes B(H)$ is invariant under $\iota \otimes t$, we have

$$\rho \circ (\iota \otimes t)(a \otimes b) = \text{Tr}(h(\iota \otimes t)(a \otimes b)) = \text{Tr}(\iota \otimes t(h)(a \otimes b)).$$

Since $\rho \circ \iota \otimes t \geq 0$ it follows that $\iota \otimes t(h) \geq 0$.

Conversely, if (ii) holds then by the above computation $\rho \circ (\iota \otimes t) \geq 0$, hence (i) holds. The equivalence of (ii) and (iii) follows since $t \otimes \iota(h) = t \otimes t(\iota \otimes t(h))$, and the fact that $t \otimes t$ is an order-automorphism.

We have

$$\begin{aligned} P(B(K), \text{copos}(H)) &= \{x \in B(K) \otimes B(H) : \iota \otimes \phi(x) \geq 0 \ \forall \text{ copositive } \phi\} \\ &= \{x \in B(K) \otimes B(H) : \iota \otimes t(x) \geq 0\}, \end{aligned}$$

because a copositive map is the composition of a completely positive map and the transpose map. Thus (ii) is equivalent to (iv), completing the proof.

Let A be a C^* -algebra. Then a map $\phi \in B(A, H)$ is called *decomposable* if it is the sum of a completely positive map and a copositive map. Otherwise ϕ is called *nondecomposable*. Since a map $\phi \in B(A, \mathbb{C}^n)$ is completely positive if and only if $\iota \otimes \phi: M_n \otimes A \rightarrow M_n \otimes M_n$ is positive [6], Lemma 5.1.3, it follows from [13] that $\phi \in B(A, \mathbb{C}^n)$ is decomposable if and only if whenever h and $t \otimes \iota(h)$ belong to $(M_n \otimes A)^+$ then $\iota \otimes \phi(h) \geq 0$. Thus ϕ is nondecomposable if and only if there exists $h \in (M_n \otimes A)^+$ such that $t \otimes \iota(h) \geq 0$ while $\iota \otimes \phi(h)$ is not positive. Suppose that $A = B(H)$, ϕ normal, and h as above. Then there exists by normality of ϕ a finite dimensional projection $f \in B(H)$ such that $\iota \otimes \phi((1 \otimes f)h(1 \otimes f))$ is not positive. We can thus assume h is of finite rank. Normalizing h we thus have by Theorem 8 that the state $\rho(x) = \text{Tr}(hx)$ satisfies the Peres condition, while by Theorem 7 ρ is entangled. We have thus proved

Theorem 9 *Let $\phi: B(H) \rightarrow M_n$ be normal positive and nondecomposable. Then there exists a normal state ρ on $B(H) \otimes M_n$ with density operator h such that $t \otimes \iota(h) \geq 0$, while $\iota \otimes \phi(h)$ is not positive. Hence ρ is entangled but satisfies the Peres condition.*

An explicit example of the situation in the above theorem is given in [13] and [5]. Then $\dim H = n = 3$, and $\phi: M_3 \rightarrow M_3$ is the nondecomposable Choi map [4]. Other examples can be found in [7] and [8]. A large class of nondecomposable maps are the projections onto spin factors of dimension greater than 6,

or more generally, positive projections onto nonreversible Jordan algebras, see [12]. See [15] for another class of nondecomposable maps. Another result close to the above theorem can be found in [2]. Previous examples of entangled states which satisfy the Peres condition have been exhibited by P.Horodecki [10].

3 Definite sets

If A and B are C^* -algebras, and $\phi: A \rightarrow B$ is a positive map of norm ≤ 1 then the (self-adjoint) *definite set* D_ϕ of ϕ is the set of self-adjoint operators in A such that $\phi(a^2) = \phi(a)^2$. If $a \in D_\phi$ and $b \in A$ then $\phi(ab + ba) = \phi(a)\phi(b) + \phi(b)\phi(a)$ and $\phi(aba) = \phi(a)\phi(b)\phi(a)$, see [12]. We show in the present section that if ϕ is of the form $\phi(x) = \sum \omega_i(x)b_i$ as in Theorem 2, then $\phi(D_\phi)$ is contained in the center of the C^* -algebra generated by $\phi(A)$. In particular, if ϕ is faithful, then D_ϕ is abelian. As a consequence we get a decomposition result for separable states.

Theorem 10 *Let A be a unital C^* -algebra and $\phi \in B(A, H)^+$ with $\phi(1) = 1$. Suppose ϕ is of the form $\phi(x) = \sum_{i=1}^n \omega_i(x)b_i$ with ω_i states of A and $b_i \in B(H)^+$. Let e be a projection in the definite set D_ϕ of ϕ , and put $f = 1 - e$. Then $\phi(e)$ and $\phi(f)$ are projections in $B(H)$ and satisfy*

$$\phi(x) = \phi(exe) + \phi(fxf) = \phi(e)\phi(x)\phi(e) + \phi(f)\phi(x)\phi(f)$$

for all $x \in A$. Hence $\phi(D_\phi)$ is an abelian C^* -algebra contained in the center of the von Neumann algebra generated by $\phi(A)$. In particular, if ϕ is faithful then D_ϕ is an abelian C^* -algebra.

Proof. Since $e \in D_\phi$, $\phi(e)$ and $\phi(f)$ are mutually orthogonal projections. Thus

$$0 = \text{Tr}(\phi(e)\phi(f)) = \text{Tr}(\sum \omega_i(e)b_i\omega_j(f)b_j) = \sum \omega_i(e)\omega_j(f)\text{Tr}(b_ib_j).$$

Since each summand is positive we have

$$\omega_i(e)\omega_j(f)\text{Tr}(b_ib_j) = 0 \quad \forall i, j.$$

In particular

$$\omega_i(e)\omega_i(f)\text{Tr}(b_i^2) = 0 \quad \forall i.$$

Since $b_i \neq 0$ either $\omega_i(e) = 0$ or $\omega_i(f) = 0$ for all i . In particular, e or f belongs to the left and right kernel of ω_i , hence $\omega_i(efx) = \omega_i(fxe) = 0$ for all x . Thus $\omega_i(x) = \omega_i(exe) + \omega_i(fxf)$ for all x , so that

$$\phi(x) = \phi(exe) + \phi(fxf) = \phi(e)\phi(x)\phi(e) + \phi(f)\phi(x)\phi(f),$$

where the last equality follows since $e, f \in D_\phi$.

To show the last statement in the theorem we consider the ultra-weakly continuous extension ϕ^{**} of ϕ to the second dual A^{**} of A . If $a \in D_\phi$ the abelian von Neumann algebra generated by a in A^{**} is contained in $D_{\phi^{**}}$ and

is generated by its projections. It thus suffices to show that for each projection $e \in D_\phi$, $\phi(e)$ belongs to the commutant of $\phi(A)$. But this is immediate from the above equation.

If ϕ is faithful then the restriction of ϕ to D_ϕ is an isomorphism, hence is abelian, since $\phi(D_\phi)$ is abelian. The proof is complete.

Corollary 11 *Let $A \subseteq B \subseteq B(H)$ be unital C^* -algebras with H separable. Suppose $\phi: B \rightarrow A$ is a conditional expectation. Then ϕ is separable if and only if A is abelian.*

Proof. By Corollary 3 if A is abelian then $\tilde{\phi}$ is separable. Since ϕ is a conditional expectation, the self-adjoint part of A equals the definite set D_ϕ , hence by Theorem 10 A is abelian if $\tilde{\phi}$ is separable, completing the proof.

Let $\tilde{\phi} = \sum \lambda_i \omega_i \otimes \rho_i$ be a faithful separable state on $M_n \otimes M_m$, which is a convex sum of states ω_i on M_n and ρ_i on M_m . By symmetry in M_n and M_m in Lemma 1, there exists a completely positive map $\psi: M_m \rightarrow M_n$ such that $\tilde{\phi}(a \otimes b) = \text{Tr}(a^t \psi(b))$. Then by Theorem 10 and the faithfulness of $\tilde{\phi}$, D_ϕ and D_ψ are abelian C^* -algebras. Let $(e_j)_{j=1,\dots,p}$ be minimal projections in D_ϕ and $(f_k)_{k=1,\dots,q}$ be minimal projections in D_ψ . From the proof of Theorem 10 the values of $\omega_i(e_j)$ and $\rho_i(f_k)$ are 0 or 1. In particular, the supports of ω_i and ρ_i are contained in some e_j and f_k respectively. Hence $e_j \otimes f_k$ are mutually orthogonal projections with sum 1 such that

$$\tilde{\phi}(x) = \sum_{i,j} \tilde{\phi}(e_j \otimes f_k x e_j \otimes f_k),$$

for all $x \in M_n \otimes M_m$

We say $\tilde{\phi}$ is *irreducible* if $D_\phi = D_\psi = \mathbb{R}$ when we have cut down by the support of $\tilde{\phi}$, and we say a family (η_i) of states are *orthogonal* if their supports are mutually orthogonal. Summing up we have shown

Corollary 12 *Every separable state on $M_n \otimes M_m$ is a convex sum of orthogonal irreducible separable states.*

References

- [1] E.M.Alfsen and F.W.Shultz, *State spaces of operator algebras*, Mathematics: Theory and Applications, Birkhauser, Boston (2001).
- [2] F.Benatti, F.Floresanini, and M.Piani, *Non-decomposable quantum semi-groups and bound entangled states*, quant-ph/0411095.
- [3] M-D. Choi, *Completely positive maps on complex matrices*, Linear Algebra and Appl. 10 (1975), 285-290.
- [4] M-D. Choi, *Positive definite biquadratic forms*, Linear Algebra and Appl. 12 (1975), 95-100.

- [5] M-D. Choi, *Positive linear maps*, AMS. Proc. Sympos. Pure Math. 1982.
- [6] E.G. Effros and Z-J. Ruan, *Operator spaces*, London Math. Soc. Monographs, New Series, 23 (2000), Oxford Press - Oxford.
- [7] K-C. Ha, S-H. Kye, and Y.S. Park, *Entangled states with positive partial transposes arising from indecomposable positive linear maps*, Phys. Lett. A 313 (2003), 163-174.
- [8] K-C. Ha and S-H. Kye, *Construction of entangled states with positive partial transposes based on indecomposable positive linear maps*, Phys. Lett. A 325 (2004), 315-323.
- [9] M. Horodicki, P.Horodicki, and R.Horodicki, *Separability of mixed states: necessary and sufficient conditions*, Physics Letters, A 223, (1996), 1-8.
- [10] P. Horodicki, *Separability criterion and inseparable mixed states with positive partial transposition*, Physics Letters, A 232, (1997), 333-339.
- [11] P. Horodicki, P.W.Shor, and M.B.Ruskai, *Entanglement breaking channels*, quant-ph /0302031.
- [12] E. Størmer, *Decomposition of positive projections on C^* -algebras*, Math. Ann. 247 (1980).
- [13] E. Størmer, *Decomposable positive maps on C^* -algebras*, Proc.A.M.S. 86, No.3 (1982), 402-404.
- [14] E. Størmer, *Extension of positive maps into $B(H)$* , Jour. Funct. Anal. 66, No.2 (1986), 235-254.
- [15] B.M.Terhal, *A family of indecomposable positive linear maps based on entangled quantum states*, quant-ph/9810091.

Department of Mathematics, University of Oslo, 0316 Oslo, Norway.
e-mail: erlings@math.uio.no